Lecture no 18 & 19

The Fundamental Theorems of Calculus

We now know enough about de nite integrals to give precise formulations of the Fundamental Theorems of Calculus. We will also look at some basic examples of these theorems in this set of notes. The next set of notes will consider some applications of these theorems.

The First Fundamental Theorem of Calculus. If f(x) is continuous on [a; b] and F (x) is any anti-derivative of f(x) on [a; b], then

$$\int_a^b f(x)\,dx = F(b)\ \ F(a):$$

In words: In order to compute a de nite integral f(x), it su ces to nd an antiderivative of f(x), then compute the di erence of this antiderivative.

The Second Fundamental Theorem of Calculus. If f(x) is continuous on an interval and a is any number in this interval, then the function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f(x). In other words,

$$\frac{d}{dx}\int_a^x \qquad :$$

In words: In order to constuct an antiderivative of a function f(x), it is enough to be able to compute the de nite integral of f(x).

Both of these theorems establish a relationship between antiderivatives (i.e., inde nite integrals) and net areas (i.e., de nite integrals.) In e ect, the First Fundamental Theorem says that de nite integra-tion reverses di erentiation, whereas the Second Fundamental Theorem says that di erentiation reverses de nite integration. Thus, the Fundamental Theorems say that de nite integration is actually a sort of antidi erentiation. This explains why we use integration notation and terminology (inde nite integrals) for antiderivatives.

From a mathematical point of view, the Second Fundamental Theorem is the \true" fundamental the-orem of calculus. The First Fundamental Theorem is just a logical consequence of Second Fundamental Theorem. However, the First Fundamental Theorem is what most non-mathematicians think of as the fundamental theorem. This is because the First Fundamental Theorem is the one that appears in appli-cations in lots of di erent subjects. Thus, the First Fundamental Theorem is of practical interest whereas the Second Fundamental Theorem is primarily of theoretical interest, although it does have some practical applications.

1. Using the First Fundamental Theorem of Calculus

Example 1: Find

$$\int_{\frac{}{2}}^{\frac{3}{2}} \sin \quad d :$$

Solution: We already know this integral is 0 by the previous set of notes. But lets compute this using the First Fundamental Theorem. First, we need to nd an antiderivative of sin : But

$$\int \sin \quad d = \quad \cos \quad + C;$$

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so we'll use F ( ) = cos in the fundamental theorem. (The fundamental theorem only requires that we nd some antiderivative, not the most general antiderivative, so we don't need to use the +C.) So,

$$Z\ {}^{3}_{2}\ \sin\ d = F(\ \tfrac{2}{3}\ )\quad F(\ )= \ \tfrac{2}{}\ \cos(\tfrac{3}{}\ ) \quad \tfrac{2}{}\ (\ \cos(\ )) = 0 + 0 = 0;$$

consistent with the answer we obtained in the last set of notes.

We make two comments on the above example. First, what if we used a di erent antiderivative? Could that have changed our answer? Fortunately the answer is no. The most general antiderivative is F ( ) = cos + C, so if we used this we would have had

$$Z\ {}^{3}_{2}\ \sin\ d = F(\ \tfrac{2}{3}\ )\quad F(\ )=(\ \tfrac{2}{}\ \cos(\tfrac{3}{}\ ) + C) \quad \tfrac{2}{}\ (\ \cos(\ ) + C) = 0 + C + 0\ \ C = 0:$$

The C's will always cancel o like this, so we don't need the +C when computing de nite integrals.

Second, we will nd it convenient to write

$$F(x)\ j^{b}{}_{a} = F(b) \qquad F(a):$$

Thus, we could write

$$Z\ {}^{3}_{\overline{2}}\ \sin\ d = F(\ )\ j\ {}^{\frac{3}{2}}_{\overline{2}} = ::::$$

Example 2: Find

$$1\ \ x\ dx:p$$

Solution: First we nd an antiderivative:

$$Z\ p\ \frac{1}{x}\ dx = Z\ x^{1=2}\ dx = 2x^{1=2} + C:$$

We'll use 2 p x as our antiderivative. Thus

$$Z\ {}^{4}_{1}\ p\ \frac{1}{x}\ dx = 2\ {}^{p}x\ j1^{4} = 2^{p}4\ \ 2^{p}1 = 2:$$

Example 3: Find

$$Z\ {}^{2}_{1}\ (x^{3} + 5x)\ dx:$$

Solution: Often times, we will just compute the antiderivatives in our heads. Thus

$$Z\ 2(x^{3} + 5x)\ dx = (4x^{4}\ \tfrac{1}{} + \tfrac{5}{}2x^{2})\ j^{2}\ \ 1 = (4\ \tfrac{1}{}\ 2^{4} + \tfrac{5}{}2\ \ 2^{2})\ (4\ \tfrac{1}{}\ (\ 1)^{4} + \tfrac{5}{}2\ (\ 1)^{2}) = 11:25$$

Example 4: Things can go wrong if the integrand has a vertical asymptote. (Please note the Fundamental Theorem does require f(x) to be de ned and continuous on the whole interval of integration.) For example, the following calculation is wrong.

$$Z\ 1\ \ x^{2}\ dx = x^{1}\ j^{1}\ \ 1 = (1)^{1}\ ((1)^{1}) = 2:$$

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We can tell immediately that something is wrong since $\int_{1}^{1} x^{2}\, dx$ is the net area of a gure that is always above the x axis, and therefore the integral better be postive. Furthermore, using approximating sums, it looks as though the integral is in nite. Thus, while the fundamental theorem gives us a very powerful method to compute de nite integrals, we do need to be careful and check that our function satis es the hypotheses of the fundamental theorem before using the conclusion of that theorem.

Having said that, I should point out that, although as stated the fundamental theorem requires the function to be continuous, in actuality the fundamental theorem holds for a class of functions more general than the continuous functions. In particular, if the integrand has a nite number of jump discontinuities then the Fundamental Theorem still works. Integrals whose integrands contain asymptotes are referred to as improper integrals and are studied in detail in Calculus 2. A lot of integrals that arise in practice (especially in Probability Theory and Quantum Mechanics) are improper integrals.

2. Using the Second Fundamental Theorem of Calculus

A large part of the di culty in understanding the Second Fundamental Theorem of Calculus is getting a grasp on the function $\int_a^x f(t)\,dt$:

As a de nite integral, we should think of A(x) as giving the net area of a geometric gure. The function A(x) depends on three di erent things. First, it depends on the integrand f(t); di erent integrand gives rise to a di erent A(x). In terms of areas, the integrand determines the height of the gure. Next, A(x) depends on a. a tells us where the rgure starts. Finally, A(x) depends on x, and x tells us where the gure stops. Now, we usually think of f(t) and a as being xed, and x is our variable. For example, if f(t) = t and a = 1; then A(x) gives us the area of a trapezoid, but the base of this trapezoid is not necessarily given. See the gures below. Thus, A(1) = 0 since the \trapezpoid" degenerates into a vertical line. A(2) is the area of a trapezoid with base 2 1 = 1 and heights 1 and 2, thus its area is $(2-1)\frac{1+2}{2}=\frac{3}{2}$:

Generally, A(x) is a trapezoid with base x 1 and heights 1 and x, so

$$A(x) = (x-1)\frac{x+1}{2} = \frac{x^2-1}{2}:$$

Now we can easily compute the derivative of A(x):

$$A^0(x) = \frac{2x}{2} = x:$$

This required quite a bit of work. First, we had to write down a formula for A(x), then we had to take the derivative. The Second Fundamental Theorem tells us that we didn’t actually need to nd an explicit formula for A(x), that we could immediately write down $A^0(x) = x$:

We remind ourselves of the Second Fundamental Theorem.

The Second Fundamental Theorem of Calculus. If f(x) is continuous on an interval and a is any number in that interval, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x):$$

In words: Integrating f(t) up to x, then computing the derivative of this with respect to x is the same as evaluating the integrand at x, i.e., the same as plugging x into f(t):

We turn to some more examples.

Example 5: Find

$$\int^{x} d$$

Solution: By the second Fundamental Theorem of Calculus, we need only evaluate the integrand, $t^2 + 3t$, at x. Hence

$$\frac{d}{dx}\int_{1}^{x} (t^2 + 3t)\, dt = x^2 + 3x:$$

(You could just as well have rst computed the de nite integral using the rst Fundamental Theorem, then computed the derivative. Doing this should again give $x^2 + 3x$. But doing so misses the whole point of the Second Fundamental Theorem.)

Example 6: Find

d Z x sin (2) d : dx

0

Solution: A few notes back we mentioned that sin (2) has no antiderivative in terms of \simple" functions, so it is impossible to compute this by rst computing the de nite integral, then computing the derivative. But this doesn't bother us since we know the Second Fundamental Theorem:

d Z x sin (2) d = sin

(x^2): dx 0

Matters are a bit more complicated if the limits of integration are not simply x. The most general situation allows both the upper and lower limits of integration to be functions of x:

The Second Fundamental Theorem of Calculus with arbitrary limits of integration. If f(x) is continuous and u(x) and ‘(x) are di erentiable functions, then

dx "Z ‘(x) u(x) f(t) dt# = $u^0(x)f(u(x))$ ‘$^0(x)f(‘(x))$:

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Example 7: According to this form of the Fundamental Theorem,

dx Z x d (x)^{0}e ^{x} = e^x + e ^{x}:

This has a nice geometric interpretation. As x increases, the area represented by R x x e^t dt expands both to the left and to the x rectangle of height e :

Example 8: d dx " Z$^{x^2}$ 1 cos t dt# = $(x^2)^0$ cos (x^2) $(1^2)^0$ cos (1^2) = 2x cos (x^2);

Example 9: d dx Z e^x t^3 dt = $(e^x)^0$ $(e^x)^3$ $(3x)^0$

$(3x)^3 = e^{4x}$ 81x^3:

Example 10: Let G(x) = R 1 0ln x p 2 e

Solution:

Z0

$$G^0(x) = \frac{d}{dx} \quad \ln x \quad \frac{1}{p\ 2}$$

$t_2{=}2$ dt: Find $G^0(x)$:

e $\quad t_2{=}2$ dt $= (\ln x)^0\ p^1\ 2$

$$\frac{e^{(\ln x)^2 = 2}}{x} = \frac{1}{p\ 2}\, e^{(\ln x)^2 = 2} :$$

The lower limit of integration is constant, so the $(0)^0\ p^1 2\ e^{\ 0_2=2}$ term is just 0, so we did not include that term.

We'll see lots of applications of the First Fundamental Theorem of Calculus in the next set of notes. Direct applications of the Second Fundamental Theorem are a bit harder to come by, but applications do exist for it. In particular, the last example is a typical sort of calculation you might do in a probability class (speci cally, this example relates the probability densities of the normal distribution and the log-normal distribution.)

3. Theory

We look at some of the more theoretical aspects of the Fundamental Theorems in this section. We begin by outlining a proof of the fundamental theorems. Actually, this argument should be very familiar to you by now. We've gone through this argument in several specific cases already.

Even though I'm taking the time to go through the proofs of these theorems, you will not be responsible for either of these proofs. Being able to do calculutions like the examples at the beginning of the set of notes on Areas and Antiderivatives is far more important than memorizing this proof. After you've done enough examples like those you should see that the proof of the fundamental theorem is the exact same argument just written out in abstract generality.

We begin with the Second Fundamental Theorem: We suppose f(x) is continuous on an interval, a is in the interval, and we need to show that the function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f(t): In other words, we need to show that $A'(x) = f(x)$; or

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x):$$

We don't have an explicit formula for the integral, so our only hope for computing the derivative is to write out the definition of derivative:

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right):$$

The numerator is the difference between two net areas, the first from a to x + h, the second from a to x. Thus, the difference is the net area from x to x + h, i.e.,

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt:$$

If h is very small then this last integral is the net area of a figure that is essentially a rectangle with width h and height f(x), so

$$\int_x^{x+h} f(t)\,dt \approx f(x)\cdot h:$$

Piecing together everything,

$$\frac{d}{dx}\int_a^x$$

This only shows

$$\frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \lim_{h\to 0}\frac{f(x)\,h}{h} = f(x);$$

so technically this is not a proof of the second Fundamental Theorem. To get a rigorous proof we would need to quantify just how good the approximation $\int_x^{x+h} f(t)\,dt \approx f(x)\,h$ is. Requiring $f(x)$ to be continuous is enough to guarantee that this approximation is good enough.

You should look back at the rst two examples in the the notes on Areas and Antiderivatives. The proof of the Second Fundamental Theorem of Calculus is exactly the same argument we used in those speci c examples.

Now for the rst Fundamental Theorem. Honestly, the proof I'm going to give is a bit confusing the rst time you see it. The problem is that the First Fundamental Theorem of Calculus is really just saying the same thing as the Second Fundamental Theorem, but just in a di erent way, so it may appear that we are not doing much in the proof. (There actually is something going on. To go from the Second to the First, we need to apply the +C Theorem, and the +C Theorem is a consequnce of the Mean Value

Theorem)

Thus, we assume f(x) is continuous on [a; b] and we let F (x) be an arbitrary antiderivative of f(x).

We need to show a^b f(t) dt = F (b) F (a): (I've swithced the variable of integration from an x to a t

want to use x for something else.) Do we know an antiderivative of f(x)? Yes, the Seco

only because

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Fundamental Theorem tells us that A(x) = R x f(t) dt is an antiderivative of f(x). Furthermore, since both F (x) and A(x) are antiderivatives of f(x R)a then there is a constant C so that F (x) = A(x) + C: Also,

$$A(b)\ \ A(a) = \int_a^b f(t)\,dt \qquad \int_a^a f(t)\,dt = \int_a^b f(t)\,dt;$$

R a f(t) dt F (b) F (a) = (A(b) + C) (A(a) + C) = (A(b) A(a)) + (C C) = A(b) A(a):

= 0 since this integral is the net area of a \rectangle" of height f(a) and width 0: But

Therefore

$$F(b)\, F(a) = \int_a^b f(t)\,dt$$

where F (x) is any antiderivative of f(x) on [a; b]:

It is possible to prove the First Fundamental Theorem of Calculus directly, that is, without recourse to the Second Fundamental Theorem. Such a proof requires doing careful estimates with Riemann sums. The advantage of the above proof of the Fundamental Theorems is that we did not have to deal directly with the Riemann Sums.

We close by writting down each of the Fundamental Theorems in yet another

form. The First Fundamental Theorem of Calculus. If $F^0(x)$ is continuous on [a;

b] then

$$\int_a^b F^0(x)\,dx = F(b) \qquad F(a):$$

The original statement involved two functions, f(x) and F (x); and F (x) was an antiderivative of f(x). But this just means that $F^0(x) = f(x)$, so we replace f(x) with $F^0(x)$: This restatement ts in nicely with the Second Fundamental Theorem. Whereas the Second Fundamental Theorem says that taking the derivative of an integral brings you back to the original function, this form of the First Fundamental Theorem says that taking the integral of a derivative brings you back to the original function.

The Second Fundamental Theorem of Calculus. If f(x) is continuous on an interval containing x, then

$$\lim_{h!0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x):$$

The original statement says $\frac{d}{}$ x f(t) dt = f(x). But writting out the de nition of the derivative then

using $\int_a^{x+h} f(t)\,dt \quad \int_a^{x} f(t)\,dt = \int_x^{x+h} f(t)\,dt$ gives this alternative form.
Corollary: If f(x) is continous on [a; b] then f(x) has an antiderivative.

This follows directly from the Second Fundamental Theorem. That theorem explicitly describes an antiderivative of f(x):

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